Math 173
Problem Set 4
October 8, 2018

1 Give an example of two norms on a vector space $\mathcal{V}$ that are *not* equivalent.

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2 Let $(\mathcal{V}, \|\cdot\|)$ be a normed space. Prove that the closed unit ball $B = \{x \in \mathcal{V} : \|x\| \leq 1\}$ is compact if and only if $\mathcal{V}$ is finite dimensional. *Hint:* Find a sequence in B that has no convergent subsequence; use norm equivalence.

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3 Let $A \in \mathsf{M}_n(\mathbb{R})$. Prove that the Schatten-p norm $\|A\|_p^p = \operatorname{tr}(|A|^p)$, where the operator absolute value $|A|$ is equal to the positive semidefinite square root of A^*A.

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4 Let $A \in \mathbb{C}^{m\times n}$ and decompose $A = U\Sigma V^*$ into its singular value decomposition where Σ is the diagonal matrix composed of the singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$. Write $\Sigma_k = \operatorname{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_k, 0, 0, \ldots\}$ for $k = 1, 2, \ldots, \max\{m, n\}$ and define the truncated singular value decomposition $A_k = U\Sigma_k V^*$. Prove[a] that the largest singular value of $A - A_k$ is σ_{k+1}.

[a]As we'll see later in the course, this says that the distance between A and A_k in the so-called *spectral norm* is σ_{k+1}. It turns out that this is the *best* rank-k approximation to A in both the spectral and Frobenius norms.

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5 Load the image located at `https://math173.github.io/img/dog.jpg` into a computational language of your choice. We can think of this grayscale image as a matrix $A \in \mathbb{R}^{m\times n}$. Where each element A_{ij} is the intensity of the pixel A_{ij}.

(a) Plot[a] (in decreasing order) the singular values of A.

(b) Plot the truncated SVD A_k as an image for $k = 5, 10, 15, 50$ and compare it to A.

(c) How many entries do we need to store to represent A_k for general $A \in \mathbb{R}^{m\times n}$? Using this, interpret A_k as a lossy compression scheme for a matrix A.

[a]You can use an SVD function. No need to implement it.

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Extra (Not Optional) Write a short (less than half a page, more than a quarter) but thoughtful proposal for your midterm project. Be concrete about what specific problem in linear algebra (broadly defined) you'd like to investigate. Consult with the teaching staff if you'd like any assistance finding a topic.

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